

DEVIATIONS OF RIESZ PROJECTIONS OF HILL OPERATORS WITH SINGULAR POTENTIALS

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ABSTRACT. It is shown that the deviations $P_n - P_n^0$ of Riesz projections

$$P_n = \frac{1}{2\pi i} \int_{C_n} (z - L)^{-1} dz, \quad C_n = \{|z - n^2| = n\},$$

of Hill operators $Ly = -y'' + v(x)y$, $x \in [0, \pi]$, with zero and H^{-1} periodic potentials go to zero as $n \rightarrow \infty$ even if we consider $P_n - P_n^0$ as operators from L^1 to L^∞ . This implies that all L^p -norms are uniformly equivalent on the Riesz subspaces $\text{Ran } P_n$.

1. INTRODUCTION

We consider the Hill operator

$$(1.1) \quad Ly = -y'' + v(x)y, \quad x \in I = [0, \pi],$$

with a singular periodic potential v , $v(x + \pi) = v(x)$, $v \in H_{loc}^{-1}(\mathbb{R})$, i.e.,

$$v(x) = v_0 + Q'(x),$$

where

$$Q \in L_{loc}^2(\mathbb{R}), \quad Q(x + \pi) = Q(x), \quad w(0) = \int_0^\pi Q(x) dx = 0,$$

so

$$Q = \sum_{m \in 2\mathbb{Z} \setminus \{0\}} w(m) e^{imx}, \quad \|v|H^{-1}\|^2 = |v_0|^2 + \sum_{m \in 2\mathbb{Z} \setminus \{0\}} |w(m)|^2 / m^2 < \infty.$$

A. Savchuk and A. Shkalikov [13] gave thorough spectral analysis of such operators. In particular, they consider a broad class of boundary conditions (bc) – see (1.6), Theorem 1.5 there – in terms of a function y and its quasi-derivative

$$u = y' - Qy.$$

Now the natural form of periodic or antiperiodic (Per^\pm) bc is the following one:

$$(1.2) \quad Per^\pm : \quad y(\pi) = \pm y(0), \quad u(\pi) = \pm u(0)$$

If the potential v happens to be an L^2 -function these bc are identical to the classical ones (see discussion in [7], Section 6.2).

The Dirichlet bc is more simple:

$$Dir : \quad y(0) = 0, \quad y(\pi) = 0;$$

it does not require quasi-derivatives, so it is defined in the same way as for L^2 -potentials v .

In our analysis of instability zones of Hill and Dirac operators (see [5] and the comments there) we follow an approach ([9, 10, 1, 2, 3, 4]) based on Fourier Method. But in the case of singular potentials it may happen that the functions

$$u_k = e^{ikx} \quad \text{or} \quad \sin kx, \quad k \in \mathbb{Z},$$

have their L -images outside L^2 . Moreover, for some singular potentials v we have $Lf \notin L^2$ for *any smooth* (say C^2 -) nonzero function f . (For example, choose

$$v(x) = \sum_r a(r) \delta_*(x - r), \quad r \text{ rational}, r \in I,$$

with $a(r) > 0$, $\sum_r a(r) = 1$ and $\delta_*(x) = \sum_{k \in \mathbb{Z}} \delta(x - k\pi)$.)

This implies, for any reasonable bc, that the eigenfunctions $\{u_k\}$ of the free operator L_{bc}^0 are not necessarily in the domain of L_{bc} . Yet, in [6, 7] we gave a justification of the Fourier method for operators L_{bc} with H^{-1} -potentials and $bc = Per^\pm$ or Dir . Our results are announced in [6], and in [7] all technical details of justification of the Fourier method are provided.

Now, in the case of singular potentials, we want to compare the Riesz projections P_n of the operator L_{bc} , defined for large enough n by the formula

$$(1.3) \quad P_n = \frac{1}{2\pi i} \int_{C_n} (z - L_{bc})^{-1} dz, \quad C_n = \{|z - n^2| = n\},$$

with the corresponding Riesz projections P_n^0 of the free operator L_{bc}^0 (although $E_n^0 = \text{Ran}(P_n^0)$ maybe have no common nonzero vectors with the domain of L_{bc}).

The main result is Theorem 2, which claims that

$$(1.4) \quad \tilde{\tau}_n = \|P_n - P_n^0\|_{L^1 \rightarrow L^\infty} \rightarrow 0.$$

This implies a sort of quantum chaos, namely all L^p -norms on the Riesz subspaces $E_n = \text{Ran} P_n$, for $bc = Per^\pm$ or Dir , are uniformly equivalent (see Theorem 6 in Section 5).

In our analysis (see [5]) of the relationship between smoothness of a potential v and the rate of decay of spectral gaps and spectral triangles a statement similar to (1.4)

$$(1.5) \quad \tau_n = \|P_n - P_n^0\|_{L^2 \rightarrow L^\infty} \rightarrow 0.$$

was crucial when we used the deviations of Dirichlet eigenvalues from periodic or anti-periodic eigenvalues to estimate the Fourier coefficients of the

potentials v . But if $v \in L^2$ it was "easy" (see [2], Section 3, Prop.4, or [5], Prop.11). Moreover, those are strong estimates: for $n \geq N(\|v\|_{L^2})$

$$(1.6) \quad \tau_n \leq \frac{C}{n} \|v\|_{L^2},$$

where C is an absolute constant. Therefore, in (1.6) only the L^2 -norm is important, so $\tau_n \leq CR/n$ holds for every v in an L^2 -ball of radius R .

Just for comparison let us mention the same type of question in the case of 1D periodic Dirac operators

$$MF = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{dF}{dx} + \begin{pmatrix} 0 & p \\ q & 0 \end{pmatrix} F, \quad 0 \leq x \leq \pi,$$

where p and q are L^2 -functions and $F = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$. The boundary conditions under consideration are Per^\pm and Dir , where

$$Per^\pm : F(\pi) = \pm F(0), \quad Dir : f_1(0) = f_2(0), \quad f_1(\pi) = f_2(\pi).$$

Then (see [11] or [5], Section 1.1)

$$E_n^0 = \left\{ \begin{pmatrix} ae^{-inx} \\ be^{inx} \end{pmatrix} : a, b \in \mathbb{C} \right\}, \quad n \in \mathbb{Z},$$

where n is even if $bc = Per^+$ and n is odd if $bc = Per^-$, and

$$E_n^0 = \{c \sin nx, c \in \mathbb{C}\}, \quad n \in \mathbb{N}$$

if $bc = Dir$. Then for

$$Q_n = \frac{1}{2\pi i} \int_{C_n} (\lambda - L)^{-1} d\lambda, \quad C_n = \{\lambda : |\lambda - n| = 1/4\},$$

we have

$$\rho_n(V) := \|Q_n - Q_n^0\|_{L^2 \rightarrow L^\infty} \rightarrow 0;$$

moreover, for any compact set $K \subset L^2$ and $V \in K$, i.e., $p, q \in K$ one can construct a sequence $\varepsilon_n(K) \rightarrow 0$ such that $\rho_n(V) \leq \varepsilon_n(K)$, $V \in K$. This has been proven in [11], Prop.8.1 and Cor.8.6; see Prop. 19 in [5] as well.

Of course, the norms τ_n in (1.5) are larger than the norms of these operators in L^2

$$t_n = \|P_n - P_n^0\|_{L^2 \rightarrow L^2} \leq \tau_n$$

and better (smaller) estimates for t_n are possible. For example, A. Savchuk and A. Shkalikov proved ([13], Sect.2.4) that $\sum t_n^2 < \infty$. This implies (by Bari-Markus theorem – see [8], Ch.6, Sect.5.3, Theorem 5.2) that the spectral decompositions

$$f = f_N + \sum_{n>N} P_n f$$

converge unconditionally. For Dirac operators the Bari–Markus condition is

$$\sum_{n \in \mathbb{Z}, |n| > N} \|Q_n - Q_n^0\|^2 < \infty.$$

This fact (and completeness of the system of Riesz subspaces $Ran Q_n$) imply unconditional convergence of the spectral decompositions. This has been proved in [11] under the assumption that the potential V is in the Sobolev space H^α , $\alpha > 1/2$ (see [11], Thm 8.8 for more precise statement). See further comments in Section 5 below as well.

The proof of Theorem 2, or the estimates of norms (1.4), are based on the perturbation theory, which gives the representation

$$(1.7) \quad P_n - P_n^0 = \frac{1}{2\pi i} \int_{C_n} (R(\lambda) - R^0(\lambda)) d\lambda,$$

where $R(\lambda) = (\lambda - L_{bc})^{-1}$ and $R^0(\lambda)$ are the resolvents of L_{bc} and of the free operator L_{bc}^0 , respectively. Often – and certainly in the above mentioned examples where $v \in L^2$ – one can get reasonable estimates for the norms $\|R(\lambda) - R^0(\lambda)\|$ on the contour C_n , and then by integration for $\|P_n - P_n^0\|$. But now, with $v \in H^{-1}$, we succeed to get good estimates for the norms $\|P_n - P_n^0\|$ after having integrated term by term the series representation

$$(1.8) \quad R - R^0 = R^0 V R^0 + R^0 V R^0 V R^0 + \dots$$

This integration kills or makes more manageable many terms, maybe in their matrix representation. Only then we go to the norm estimates. Technical details of this procedure (Section 3) is the core of the proof of Theorem 2, and of this paper.

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2. MAIN RESULT

By our Theorem 21 in [7] (about spectra localization), the operator $L_{Per\pm}$ has, for large enough n , exactly two eigenvalues (counted with their algebraic multiplicity) inside the disc of radius n about n^2 (periodic for even n or antiperiodic for odd n). The operator L_{Dir} has one eigenvalue in these discs for all large enough n .

Let E_n be the corresponding Riesz invariant subspace, and let P_n be the corresponding Riesz projection, i.e.,

$$P_n = \frac{1}{2\pi i} \int_{C_n} (\lambda - L)^{-1} d\lambda,$$

where $C_n = \{\lambda : |\lambda - n^2| = n\}$. We denote by P_n^0 the Riesz projector that corresponds to the free operator.

Proposition 1. *In the above notations, for boundary conditions $bc = Per^\pm$ or Dir ,*

$$(2.1) \quad \|P_n - P_n^0\|_{L^2 \rightarrow L^\infty} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

As a matter of fact we will prove a stronger statement.

Theorem 2. *In the above notations, for boundary conditions $bc = Per^\pm$ or Dir ,*

$$(2.2) \quad \|P_n - P_n^0\|_{L^1 \rightarrow L^\infty} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Proof. We give a complete proof in the case $bc = Per^\pm$. If $bc = Dir$ the proof is the same, and only minor changes are necessary due to the fact that in this case the orthonormal system of eigenfunctions of L^0 is $\{\sqrt{2} \sin nx, n \in \mathbb{N}\}$ (while it is $\{\exp(imx), m \in 2\mathbb{Z}\}$ for $bc = Per^+$, and $\{\exp(imx), m \in 1+2\mathbb{Z}\}$ for $bc = Per^-$). So, roughly speaking, the only difference is that when working with $bc = Per^\pm$ the summation indexes in our formulas below run, respectively, in $2\mathbb{Z}$ and $1+2\mathbb{Z}$, while for $bc = Dir$ the summation indexes have to run in \mathbb{N} . Therefore, we consider in detail only $bc = Per^\pm$, and provide some formulas for the case $bc = Dir$.

Let

$$(2.3) \quad B_{km}(n) := \langle (P_n - P_n^0)e_m, e_k \rangle.$$

We are going to prove that

$$(2.4) \quad \sum_{k,m} |B_{km}(n)| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Of course, the convergence of the series in (2.4) means that the operator with the matrix $B_{km}(n)$ acts from ℓ^∞ into ℓ^1 .

The Fourier coefficients of an L^1 -function form an ℓ^∞ -sequence. On the other hand,

$$(2.5) \quad D = \sup_{x,n} |e_n(x)| < \infty.$$

Therefore, the operators $P_n - P_n^0$ act from L^1 into L^∞ (even into C) and

$$(2.6) \quad \|P_n - P_n^0\|_{L^1 \rightarrow L^\infty} \leq D^2 \sum_{k,m} |B_{km}(n)|.$$

Indeed, if $\|f\|_{L^1} = 1$ and $f = \sum f_m e_m$, then $|f_m| \leq D$ and

$$(P_n - P_n^0)f = \sum_k \left(\sum_m B_{km} f_m \right) e_k.$$

Taking into account (2.5), we get

$$\|(P_n - P_n^0)f\|_{L^\infty} \leq D \sum_k \left| \sum_m B_{km} f_m \right| \leq D^2 \sum_k \sum_m |B_{km}|,$$

which proves (2.6).

In [7], Section 5, we gave a detailed analysis of the representation

$$R_\lambda - R_\lambda^0 = \sum_{s=0}^{\infty} K_\lambda (K_\lambda V K_\lambda)^{s+1} K_\lambda,$$

where $K_\lambda = \sqrt{R_\lambda^0}$ – see [7], (5.13-14) and what follows there. By (1.7),

$$P_n - P_n^0 = \frac{1}{2\pi i} \int_{C_n} \sum_{s=0}^{\infty} K_\lambda (K_\lambda V K_\lambda)^{s+1} K_\lambda d\lambda.$$

if the series on the right converges. Thus

$$(2.7) \quad \langle (P_n - P_n^0)e_m, e_k \rangle = \sum_{s=0}^{\infty} \frac{1}{2\pi i} \int_{C_n} \langle K_\lambda (K_\lambda V K_\lambda)^{s+1} K_\lambda e_m, e_k \rangle d\lambda,$$

so we have

$$(2.8) \quad \sum_{k,m} |\langle (P_n - P_n^0)e_m, e_k \rangle| \leq \sum_{s=0}^{\infty} A(n, s),$$

where

$$(2.9) \quad A(n, s) = \sum_{k,m} \left| \frac{1}{2\pi i} \int_{C_n} \langle K_\lambda (K_\lambda V K_\lambda)^{s+1} K_\lambda e_m, e_k \rangle d\lambda \right|.$$

By the matrix representation of the operators K_λ and V (see more details in [7], (5.15-22)) it follows that

$$(2.10) \quad \langle K_\lambda (K_\lambda V K_\lambda) K_\lambda e_m, e_k \rangle = \frac{V(k-m)}{(\lambda - k^2)(\lambda - m^2)}, \quad k, m \in n + 2\mathbb{Z},$$

for $bc = Per^\pm$, and

$$(2.11) \quad \langle K_\lambda (K_\lambda V K_\lambda) K_\lambda e_m, e_k \rangle = \frac{|k-m|\tilde{q}(|k-m|) - (k+m)\tilde{q}(k+m)}{\sqrt{2}(\lambda - k^2)(\lambda - m^2)}, \quad k, m \in \mathbb{N},$$

for $bc = Dir$. Let us remind that $\tilde{q}(m)$ are the sine Fourier coefficients of the function $Q(x)$, i.e.,

$$Q(x) = \sum_{m=1}^{\infty} \tilde{q}(m) \sqrt{2} \sin mx.$$

The matrix representations of $K_\lambda(K_\lambda V K_\lambda)K_\lambda$ in (2.10) and (2.8) are the "building blocks" for the matrices of the products of the form $K_\lambda(K_\lambda V K_\lambda)^s K_\lambda$ that we have to estimate below. For convenience, we set

$$(2.12) \quad V(m) = mw(m), \quad w \in \ell^2(2\mathbb{Z}), \quad r(m) = \max(|w(m)|, |w(-m)|)$$

if $bc = Per^\pm$, and

$$(2.13) \quad \tilde{q}(0) = 0, \quad r(m) = \tilde{q}(|m|), \quad m \in \mathbb{Z}.$$

if $bc = Dir$. We use the notations (2.12) in the estimates related to $bc = Per^\pm$ below, and if one would use in a similar way (2.13) in the Dirichlet case, then the corresponding computations becomes practically identical (the only difference will be that in the Dirichlet case the summation will run over \mathbb{Z}). So, further we consider only the case $bc = Per^\pm$.

Let us calculate the first term on the right-hand side of (2.7) (i.e., the term coming for $s = 0$). We have

$$(2.14) \quad \frac{1}{2\pi i} \int_{C_n} \frac{V(k-m)}{(\lambda-k^2)(\lambda-m^2)} d\lambda = \begin{cases} \frac{V(k-n)}{(n^2-k^2)} & m = \pm n, \quad k \neq \pm n, \\ \frac{V(\pm n-m)}{(n^2-m^2)} & k = \pm n, \quad m \neq \pm n, \\ 0 & \text{otherwise.} \end{cases}$$

Thus

$$\begin{aligned} A(n, 0) &= \sum_{k, m} \left| \frac{1}{2\pi i} \int_{C_n} \langle K_\lambda(K_\lambda V K_\lambda) K_\lambda e_m, e_k \rangle \right| \\ &= \sum_{k \neq \pm n} \frac{|V(k-n)|}{|n^2-k^2|} + \sum_{k \neq \pm n} \frac{|V(k+n)|}{|n^2-k^2|} + \sum_{m \neq \pm n} \frac{|V(-n+m)|}{|n^2-m^2|} + \sum_{m \neq \pm n} \frac{|V(n-m)|}{|n^2-m^2|}. \end{aligned}$$

By the Cauchy inequality, we estimate the first sum on the right-hand side:

$$\begin{aligned} (2.15) \quad \sum_{k \neq \pm n} \frac{|V(k-n)|}{|n^2-k^2|} &= \sum_{k \neq \pm n} \frac{|k-n||w(k-n)|}{|n^2-k^2|} \\ &\leq \sum_{k \neq -n} \frac{r(k-n)}{|n+k|} \leq \sum_{k>0} \dots + \sum_{k \leq 0, k \neq -n} \dots \\ &\leq \left(\sum_{k>0} \frac{1}{|n+k|^2} \right)^{1/2} \cdot \|r\| + \left(\sum_{k \leq 0, k \neq -n} \frac{1}{|n+k|^2} \right)^{1/2} \left(\sum_{k \leq 0} (r(n-k))^2 \right)^{1/2} \\ &\leq \frac{\|r\|}{\sqrt{n}} + \mathcal{E}_n(r). \end{aligned}$$

Since each of the other three sums could be estimated in the same way, we get

$$(2.16) \quad A(n, 0) \leq \sum_{k,m} \left| \frac{1}{2\pi i} \int_{C_n} \langle K_\lambda(K_\lambda V K_\lambda) K_\lambda e_m, e_k \rangle d\lambda \right| \leq \frac{4\|r\|}{\sqrt{n}} + 4\mathcal{E}_n(r).$$

Next we estimate $A(n, s)$, $s \geq 1$. By the matrix representation of K_λ and V – see (2.10) – we have

$$(2.17) \quad \langle K_\lambda(K_\lambda V K_\lambda)^{s+1} K_\lambda e_m, e_k \rangle = \frac{\Sigma(\lambda; s, k, m)}{(\lambda - k^2)(\lambda - m^2)}$$

where

$$(2.18) \quad \Sigma(\lambda; s, k, m) = \sum_{j_1, \dots, j_s} \frac{V(k - j_1)V(j_1 - j_2) \cdots V(j_{s-1} - j_s)V(j_s - m)}{(\lambda - j_1^2)(\lambda - j_2^2) \cdots (\lambda - j_s^2)},$$

$k, m, j_1, \dots, j_s \in n + 2\mathbb{Z}$. For convenience, we set also

$$(2.19) \quad \Sigma(\lambda; 0, k, m) = V(k - m).$$

In view of (2.9), we have

$$(2.20) \quad A(n, s) = \sum_{k,m} \left| \frac{1}{2\pi i} \int_{C_n} \frac{\Sigma(\lambda; s, k, m)}{(\lambda - k^2)(\lambda - m^2)} d\lambda \right|.$$

Let us consider the following sub-sums of $\Sigma(\lambda; s, k, m)$:

$$(2.21) \quad \Sigma^0(\lambda; s, k, m) = \sum_{j_1, \dots, j_s \neq \pm n} \cdots \quad \text{for } s \geq 1, \quad \Sigma^0(\lambda; 0, k, m) := V(k - m);$$

$$(2.22) \quad \Sigma^1(\lambda; s, k, m) = \sum_{\exists \text{ one } j_\nu = \pm n} \cdots \quad \text{for } s \geq 1;$$

$$(2.23) \quad \Sigma^*(\lambda; s, k, m) = \sum_{\exists j_\nu = \pm n} \cdots, \quad \Sigma^{**}(\lambda; s, k, m) = \sum_{\exists j_\nu, j_\mu = \pm n} \cdots, \quad s \geq 2$$

(i.e., Σ^0 is the sub-sum of Σ over those indices j_1, \dots, j_s that are different from $\pm n$, in Σ^1 exactly one summation index is equal to $\pm n$, in Σ^* at least one summation index is equal to $\pm n$, and in Σ^{**} at least two summation indices are equal to $\pm n$). Notice that

$$\Sigma(\lambda; s, k, m) = \Sigma^0(\lambda; s, k, m) + \Sigma^*(\lambda; s, k, m), \quad s \geq 1,$$

and

$$\Sigma(\lambda; s, k, m) = \Sigma^0(\lambda; s, k, m) + \Sigma^1(\lambda; s, k, m) + \Sigma^{**}(\lambda; s, k, m), \quad s \geq 2.$$

In these notations we have

$$(2.24) \quad \sum_{m,k \neq \pm n} \left| \frac{1}{2\pi i} \int_{C_n} \frac{\Sigma^0(\lambda; s, k, m)}{(\lambda - k^2)(\lambda - m^2)} d\lambda \right| = 0$$

because, for $m, k \neq \pm n$, the integrand is an analytic function of λ in the disc $\{\lambda : |\lambda - n^2| \leq n/4\}$.

Therefore, $A(n, s)$ could be estimated as follows:

$$(2.25) \quad A(n, 1) \leq \sum_{i=1}^5 A_i(n, 1),$$

and

$$(2.26) \quad A(n, s) \leq \sum_{i=1}^7 A_i(n, s), \quad s \geq 2,$$

where

$$(2.27) \quad A_1(n, s) = \sum_{k,m=\pm n} n \cdot \sup_{\lambda \in C_n} \left| \frac{\Sigma(\lambda; s, k, m)}{(\lambda - k^2)(\lambda - m^2)} \right|,$$

$$(2.28) \quad A_2(n, s) = \sum_{k=\pm n, m \neq \pm n} n \cdot \sup_{\lambda \in C_n} \left| \frac{\Sigma^0(\lambda; s, k, m)}{(\lambda - k^2)(\lambda - m^2)} \right|,$$

$$(2.29) \quad A_3(n, s) = \sum_{k=\pm n, m \neq \pm n} n \cdot \sup_{\lambda \in C_n} \left| \frac{\Sigma^*(\lambda; s, k, m)}{(\lambda - k^2)(\lambda - m^2)} \right|,$$

$$(2.30) \quad A_4(n, s) = \sum_{k \neq \pm n, m=\pm n} n \cdot \sup_{\lambda \in C_n} \left| \frac{\Sigma^0(\lambda; s, k, m)}{(\lambda - k^2)(\lambda - m^2)} \right|,$$

$$(2.31) \quad A_5(n, s) = \sum_{k \neq \pm n, m=\pm n} n \cdot \sup_{\lambda \in C_n} \left| \frac{\Sigma^*(\lambda; s, k, m)}{(\lambda - k^2)(\lambda - m^2)} \right|,$$

$$(2.32) \quad A_6(n, s) = \sum_{k,m \neq \pm n} n \cdot \sup_{\lambda \in C_n} \left| \frac{\Sigma^1(\lambda; s, k, m)}{(\lambda - k^2)(\lambda - m^2)} \right|,$$

$$(2.33) \quad A_7(n, s) = \sum_{k,m \neq \pm n} n \cdot \sup_{\lambda \in C_n} \left| \frac{\Sigma^{**}(\lambda; s, k, m)}{(\lambda - k^2)(\lambda - m^2)} \right|.$$

First we estimate $A_1(n, s)$. By (2.10) and [7], Lemma 19 (inequalities (5.30), (5.31)),

$$(2.34) \quad \sup_{\lambda \in C_n} \|K_\lambda\| = \frac{2}{\sqrt{n}}, \quad \sup_{\lambda \in C_n} \|K_\lambda V K_\lambda\| \leq \rho_n := C \left(\frac{\|r\|}{\sqrt{n}} + \mathcal{E}_{\sqrt{n}}(r) \right),$$

where $r = (r(m))$ is defined by the relations (2.12) and C is an absolute constant.

Lemma 3. *In the above notations*

$$(2.35) \quad \sup_{\lambda \in C_n} \left| \frac{\Sigma(\lambda; s, k, m)}{(\lambda - k^2)(\lambda - m^2)} \right| \leq \frac{1}{n} \rho_n^{s+1}.$$

Proof. Indeed, in view of (2.18) and (2.34), we have

$$\begin{aligned} \left| \frac{\Sigma(\lambda; s, k, m)}{(\lambda - k^2)(\lambda - m^2)} \right| &= |K_\lambda (K_\lambda V K_\lambda)^{s+1} K_\lambda e_k, e_m| \\ &\leq \|K_\lambda (K_\lambda V K_\lambda)^{s+1} K_\lambda\| \leq \|K_\lambda\| \cdot \|K_\lambda V K_\lambda\|^{s+1} \cdot \|K_\lambda\| \leq \frac{1}{n} \rho_n^{s+1}, \end{aligned}$$

which proves (2.35). \square

Now we estimate $A_1(n, s)$. By (2.35),

$$(2.36) \quad A_1(n, s) = \sum_{m, k = \pm n} n \cdot \sup_{\lambda \in C_n} \left| \frac{\Sigma(\lambda; s, k, m)}{(\lambda - k^2)(\lambda - m^2)} \right| \leq 4 \rho_n^{s+1}.$$

To estimate $A_2(n, s)$, we consider $\Sigma^0(\lambda; s, k, m)$ for $k = \pm n$. From the elementary inequality

$$(2.37) \quad \frac{1}{|\lambda - j^2|} \leq \frac{2}{|n^2 - j^2|} \quad \text{for } \lambda \in C_n, j \in n + 2\mathbb{Z}, j \neq \pm n,$$

it follows, for $m \neq \pm n$,

$$(2.38) \quad \begin{aligned} &\sup_{\lambda \in C_n} \left| \frac{\Sigma^0(\lambda; s, \pm n, m)}{(\lambda - n^2)(\lambda - m^2)} \right| \leq \frac{1}{n} \cdot 2^{s+1} \times \\ &\times \sum_{j_1, \dots, j_s \neq \pm n} \frac{|V(\pm n - j_1)V(j_1 - j_2) \cdots V(j_{s-1} - j_s)V(j_s - m)|}{|n^2 - j_1^2||n^2 - j_2^2| \cdots |n^2 - j_s^2||n^2 - m^2|}. \end{aligned}$$

Thus, taking the sum of both sides of (2.38) over $m \neq \pm n$, we get

$$(2.39) \quad A_2(n, s) \leq 2^{s+1} [L(s+1, n) + L(s+1, -n)],$$

where

$$(2.40) \quad L(p, d) := \sum_{i_1, \dots, i_p \neq \pm n} \frac{|V(d - i_1)|}{|n^2 - i_1^2|} \cdot \frac{|V(i_1 - i_2)|}{|n^2 - i_2^2|} \cdots \frac{|V(i_{p-1} - i_p)|}{|n^2 - i_p^2|}.$$

The roles of k and m in $A_2(n, s)$ and $A_4(n, s)$ are symmetric, so $A_4(n, s)$ could be estimated in an analogous way. Indeed, for $k \neq \pm n$, we have

$$(2.41) \quad \sup_{\lambda \in C_n} \left| \frac{\Sigma^0(\lambda; s, k, \pm n)}{(\lambda - k^2)(\lambda - n^2)} \right| \leq \frac{1}{n} \cdot 2^{s+1} \times$$

$$\times \sum_{j_1, \dots, j_s \neq \pm n} \frac{|V(k - j_1)V(j_1 - j_2) \cdots V(j_{s-1} - j_s)V(j_s - \pm n)|}{|n^2 - k^2||n^2 - j_1^2||n^2 - j_2^2| \cdots |n^2 - j_s^2|}.$$

Thus, taking the sum of both sides of (2.41) over $k \neq \pm n$, we get

$$(2.42) \quad A_4(n, s) \leq 2^{s+1} [R(s+1, n) + R(s+1, -n)],$$

where

$$(2.43) \quad R(p, d) := \sum_{i_1, \dots, i_p \neq \pm n} \frac{|V(i_1 - i_2)|}{|n^2 - i_1^2|} \cdots \frac{|V(i_{p-1} - i_p)|}{|n^2 - i_{p-1}^2|} \cdot \frac{|V(i_p - d)|}{|n^2 - i_p^2|}.$$

Below (see Lemma 4 and its proof in Sect. 3) we estimate the sums $L(p, \pm n)$ and $R(p, \pm n)$. But now we are going to show that $A_i(n, s)$, $i = 3, 5, 6, 7$, could be estimated in terms of L and R from (2.40), (2.43) as well.

To estimate $A_6(n, s)$ we write the expression $\frac{\Sigma^1(\lambda; s, k, m)}{(\lambda - k^2)(\lambda - m^2)}$ in the form

$$\sum_{\nu=1}^s \sum_{d=\pm n} \frac{1}{\lambda - k^2} \Sigma^0(\lambda; \nu - 1, k, d) \frac{1}{\lambda - n^2} \Sigma^0(\lambda; s - \nu, d, m) \frac{1}{\lambda - m^2}$$

By (2.37), the absolute values of the terms of this double sum do not exceed:

(a) for $\nu = 1$

$$2^{s+1} \cdot \frac{|V(k - \pm n)|}{|n^2 - k^2|} \cdot \frac{1}{n} \cdot \sum_{i_1, \dots, i_{s-1} \neq \pm n} \frac{|V(\pm n - i_1)||V(i_1 - i_2)| \cdots |V(i_{s-1} - m)|}{|n^2 - i_1^2| \cdots |n^2 - i_{s-1}^2||n^2 - m^2|}.$$

(b) for $\nu = s$

$$2^{s+1} \cdot \left(\sum_{i_1, \dots, i_{s-1} \neq \pm n} \frac{|V(k - i_1)||V(i_1 - i_2)| \cdots |V(i_{s-1} - \pm n)|}{|n^2 - k^2||n^2 - i_1^2||n^2 - i_2^2| \cdots |n^2 - i_{s-1}^2|} \right) \cdot \frac{1}{n} \cdot \frac{|V(\pm n - m)|}{|n^2 - m^2|}$$

(c) for $1 < \nu < s$

$$2^{s+1} \cdot \left(\sum_{i_1, \dots, i_{\nu-1} \neq \pm n} \frac{|V(k - i_1)||V(i_1 - i_2)| \cdots |V(i_{\nu-1} - \pm n)|}{|n^2 - k^2||n^2 - i_1^2||n^2 - i_2^2| \cdots |n^2 - i_{\nu-1}^2|} \right) \cdot \frac{1}{n} \\ \times \sum_{i_1, \dots, i_{s-\nu} \neq \pm n} \frac{|V(\pm n - i_1)||V(i_1 - i_2)| \cdots |V(i_{s-\nu} - m)|}{|n^2 - i_1^2| \cdots |n^2 - i_{s-\nu}^2||n^2 - m^2|}.$$

Therefore, taking the sum over $m, k \neq \pm n$, we get

$$(2.44) \quad A_6(n, s) \leq 2^{s+1} \cdot \sum_{\nu=1}^s \sum_{d=\pm n} R(\nu, d) \cdot L(s+1-\nu, d).$$

One could estimate $A_3(n, s)$, $A_5(n, s)$ and $A_7(n, s)$ in an analogous way. We will write the core formulas but omit some details.

To estimate $A_3(n, s)$, we use the identity

$$\frac{\Sigma(\lambda; s, k, \pm n)}{(\lambda - k^2)(\lambda - n^2)} = \sum_{\nu=1}^s \sum_{d=\pm n} \frac{1}{\lambda - k^2} \Sigma^0(\lambda; \nu-1, k, d) \frac{1}{\lambda - n^2} \Sigma(\lambda; s-\nu, d, \pm n) \frac{1}{\lambda - n^2}.$$

In view of (2.35), (2.37) and (2.43), from here it follows that

$$(2.45) \quad A_3(n, s) \leq 2^{s+1} \cdot \sum_{\nu=1}^s \sum_{d=\pm n} R(\nu, d) \cdot \rho_n^{s-\nu+1}.$$

We estimate $A_5(n, s)$ by using the identity

$$\frac{\Sigma(\lambda; s, \pm n, m)}{(\lambda - n^2)(\lambda - m^2)} = \sum_{\nu=1}^s \sum_{d=\pm n} \frac{1}{\lambda - n^2} \Sigma(\lambda; \nu-1, \pm n, d) \frac{1}{\lambda - n^2} \Sigma^0(\lambda; s-\nu, d, m) \frac{1}{\lambda - m^2}.$$

In view of (2.35), (2.37) and (2.40), from here it follows that

$$(2.46) \quad A_5(n, s) \leq 2^{s+1} \cdot \sum_{\nu=1}^s \sum_{d=\pm n} \rho_n^\nu \cdot L(s - \nu + 1, d).$$

Finally, to estimate $A_7(n, s)$ we use the identity

$$\begin{aligned} \frac{\Sigma(\lambda; s, k, m)}{(\lambda - k^2)(\lambda - m^2)} &= \sum_{1 \leq \nu < \mu \leq s} \sum_{d_1, d_2 = \pm n} \frac{1}{\lambda - k^2} \Sigma^0(\lambda; \nu-1, k, d_1) \times \\ &\times \frac{1}{\lambda - n^2} \Sigma(\lambda; \mu - \nu - 1, d_1, d_2) \frac{1}{\lambda - n^2} \Sigma^0(\lambda; s - \mu, d_2, m) \frac{1}{\lambda - m^2} \end{aligned}$$

In view of (2.35), (2.37), (2.40) and (2.43), from here it follows that

$$(2.47) \quad A_7(n, s) \leq 2^s \cdot \sum_{1 \leq \nu < \mu \leq s} \sum_{d_1, d_2 = \pm n} R(\nu, d_1) \cdot \rho_n^{\mu-\nu} \cdot L(s - \mu + 1, d_2).$$

Next we estimate $L(p, \pm n)$ and $R(p, \pm n)$. Changing the indices in (2.43) by

$$j_\nu = -i_{p+1-\nu}, \quad 1 \leq \nu \leq p,$$

we get

$$(2.48) \quad R(p, d) = L(p, -d).$$

Therefore, it is enough to estimate only $L(p, \pm n)$.

Lemma 4. *In the above notations, there exists a sequence of positive numbers $\varepsilon_n \rightarrow 0$ such that, for large enough n ,*

$$(2.49) \quad L(s, \pm n) \leq (\varepsilon_n)^s, \quad \forall s \in \mathbb{N}.$$

The proof of this lemma is technical. It is given in detail in Section 3. Then in Section 4 we complete the proof of Theorem 2. With (2.48) and (2.49), in Section 4 we will use Lemma 4 in the following form.

Corollary 5. *In the above notations, there exists a sequence of positive numbers $\varepsilon_n \rightarrow 0$ such that, for large enough n ,*

$$(2.50) \quad \max\{L(s, \pm n), R(s, \pm n)\} \leq (\varepsilon_n)^s, \quad \forall s \in \mathbb{N}.$$

3. PROOFS AND TECHNICAL INEQUALITIES

We follow the notations from Section 2. Now we prove Lemma 4.

Proof. First we show that

$$(3.1) \quad L(s, \pm n) \leq \sigma(n, s), \quad s \geq 1,$$

where

$$(3.2) \quad \sigma(n, 1) = \sum_{j_1 \neq \pm n} \frac{r(n + j_1)}{|n^2 - j_1^2|},$$

for $s \geq 2$

$$(3.3) \quad \begin{aligned} \sigma(n, s) := & \sum_{j_1, \dots, j_s \neq \pm n} \left(\frac{1}{|n - j_1|} + \frac{1}{|n + j_2|} \right) \cdots \left(\frac{1}{|n - j_{s-1}|} + \frac{1}{|n + j_s|} \right) \frac{1}{|n - j_s|} \\ & \times r(n + j_1) r(j_1 + j_2) \cdots r(j_{s-1} + j_s), \end{aligned}$$

and the sequence $r = (r(m))$ is defined by (2.12).

For $s = 1$ we have, with $i_1 = -j_1$,

$$L(1, n) = \sum_{j_1 \neq \pm n} \frac{|V(n - j_1)|}{|n^2 - j_1^2|} = \sum_{i_1 \neq \pm n} \frac{|V(n + i_1)|}{|n^2 - i_1^2|} = \sum_{i_1 \neq \pm n} \frac{|w(n + i_1)|}{|n - i_1|} \leq \sum_{i_1 \neq \pm n} \frac{r(n + i_1)}{|n - i_1|}$$

(where (2.12) is used). In an analogous way we get

$$L(1, -n) = \sum_{j_1 \neq \pm n} \frac{|V(-n - j_1)|}{|n^2 - j_1^2|} = \sum_{i_1 \neq \pm n} \frac{|w(-n - i_1)|}{|n - i_1|} \leq \sum_{i_1 \neq \pm n} \frac{r(n + i_1)}{|n - i_1|},$$

so, (3.1) holds for $s = 1$.

Let $s \geq 2$. Changing the indices of summation in (2.40) (considered with $p = s$ and $d = n$) by $j_\nu = (-1)^\nu i_\nu$, we get

$$\begin{aligned} L(s, n) &= \sum_{j_1, \dots, j_s \neq \pm n} \frac{|V(n + j_1)|}{|n^2 - j_1^2|} \frac{|V(-j_1 - j_2)|}{|n^2 - j_2^2|} \cdots \frac{|V[(-1)^{s-1}(j_{s-1} + j_s)]|}{|j_s^2 - n^2|} \\ &= \sum_{j_1, \dots, j_s \neq \pm n} \frac{|n + j_1| |j_2 + j_1| \cdots |j_s + j_{s-1}|}{|j_1^2 - n^2| |j_2^2 - n^2| \cdots |j_s^2 - n^2|} |w(n + j_1) w(-j_1 - j_2) \cdots w[(-1)^{s-1}(j_{s-1} + j_s)]| \\ &\leq \sum_{j_1, \dots, j_s \neq \pm n} \frac{|j_2 + j_1| \cdots |j_s + j_{s-1}|}{|n - j_1| |n^2 - j_2^2| \cdots |n^2 - j_s^2|} r(n + j_1) r(j_1 + j_2) \cdots r(j_{s-1} + j_s). \end{aligned}$$

By the identity

$$\frac{i+k}{(n-i)(n+k)} = \frac{1}{n-i} - \frac{1}{n+k},$$

we get that the latter sum does not exceed

$$\begin{aligned} & \sum_{j_1, \dots, j_s \neq \pm n} \left| \frac{1}{n-j_1} - \frac{1}{n+j_2} \right| \cdots \left| \frac{1}{n-j_{s-1}} - \frac{1}{n+j_s} \right| \frac{1}{|n-j_s|} \\ & \times r(n+j_1)r(j_1+j_2) \cdots r(j_{s-1}+j_s) \leq \sigma(n, s). \end{aligned}$$

Changing the indices of summation in (2.40) (considered with $p = s$ and $d = -n$) by $j_\nu = (-1)^{\nu+1}i_\nu$, one can show that $L(s, -n) \leq \sigma(n, s)$. Since the proof is the same we omit the details. This completes the proof of (3.1).

In view of (3.1), Lemma 4 will be proved if we show that there exists a sequence of positive numbers $\varepsilon_n \rightarrow 0$ such that, for large enough n ,

$$(3.4) \quad \sigma(n, s) \leq (\varepsilon_n)^s, \quad \forall s \in \mathbb{N}.$$

In order to prove (3.4) we need the following statements.

Lemma 6. *Let $r = (r(k)) \in \ell^2(2\mathbb{Z})$, $r(k) \geq 0$, and let*

$$(3.5) \quad \sigma_1(n, s; m) = \sum_{j_1, \dots, j_s \neq n} \frac{r(m+j_1)}{|n-j_1|} \frac{r(j_1+j_2)}{|n-j_2|} \cdots \frac{r(j_{s-1}+j_s)}{|n-j_s|}, \quad n, s \in \mathbb{N},$$

where $m, j_1, \dots, j_s \in n + 2\mathbb{Z}$. Then, with

$$(3.6) \quad \tilde{\rho}_n := \mathcal{E}_n(r) + 2\|r\|/\sqrt{n},$$

we have, for $n \geq 4$,

$$(3.7) \quad \sigma_1(n, 1; m) \leq \begin{cases} \tilde{\rho}_n & \text{if } |m-n| \leq n/2, \\ \|r\| & \text{for arbitrary } m \in n + 2\mathbb{Z}, \end{cases}$$

$$(3.8) \quad \sigma_1(n, 2p; m) \leq (2\|r\|\tilde{\rho}_n)^p, \quad \sigma_1(n, 2p+1; m) \leq \|r\| \cdot (2\|r\|\tilde{\rho}_n)^p.$$

Proof. Let us recall that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \pi^2/6, \quad \sum_{k=n+1}^{\infty} \frac{1}{k^2} < \sum_{k=n+1}^{\infty} \left(\frac{1}{k-1} - \frac{1}{k} \right) = \frac{1}{n}.$$

Therefore, one can easily see that

$$\sum_{i \in 2\mathbb{Z}, i \neq 0} \frac{1}{i^2} = \pi^2/12 < 1, \quad \sum_{i \in 2\mathbb{Z}, |i| > n/2} \frac{1}{i^2} < 4/n, \quad n \geq 4.$$

By the Cauchy inequality,

$$\sigma_1(n, 1; m) = \sum_{j_1 \neq n} \frac{r(m + j_1)}{|n - j_1|} \leq \left(\sum_{j_1 \neq \pm n} |n - j_1|^{-2} \right)^{1/2} \cdot \|r\| \leq \|r\|,$$

which proves the second case in (3.7).

If $|m - n| \leq n/2$ then we have $n/2 \leq m \leq 3n/2$. Let us write $\sigma_1(n, 1; m)$ in the form

$$\sigma_1(n, 1; m) = \sum_{0 < |j_1 - n| \leq n/2} \frac{r(m + j_1)}{|n - j_1|} + \sum_{|j_1 - n| > n/2} \frac{r(m + j_1)}{|n - j_1|}$$

and apply the Cauchy inequality to each of the above sums. In the first sum $n/2 \leq j \leq 3n/2$, so $j + m \geq n$, and therefore, we get

$$\sigma_1(n, 1; m) \leq \left(\sum_{i \geq n} |r(i)|^2 \right)^{1/2} \cdot 1 + \|r\| \cdot \left(\sum_{|n - j_1| > n/2} |j_1 - n|^{-2} \right)^{1/2}.$$

Thus

$$\sigma_1(n, 1; m) \leq \mathcal{E}_n(r) + \frac{2\|r\|}{\sqrt{n}} = \tilde{\rho}_n \quad \text{if } |n - m| \leq n/2.$$

This completes the proof of (3.7).

Next we estimate $\sigma_1(n, 2; m)$. We have

$$\begin{aligned} \sigma_1(n, 2; m) &= \sum_{j_1 \neq n} \frac{r(m + j_1)}{|n - j_1|} \sum_{j_2 \neq n} \frac{r(j_1 + j_2)}{|n - j_2|} \\ &= \sum_{0 < |j_1 - n| \leq n/2} \frac{r(m + j_1)}{|n - j_1|} \cdot \sigma_1(n, 1; j_1) + \sum_{|j_1 - n| > n/2} \frac{r(m + j_1)}{|n - j_1|} \cdot \sigma_1(n, 1; j_1) \end{aligned}$$

By the Cauchy inequality and (3.7), we get

$$\sum_{0 < |j_1 - n| \leq n/2} \frac{r(m + j_1)}{|n - j_1|} \sigma_1(n, 1; j_1) \leq \|r\| \cdot \sup_{0 < |j_1 - n| \leq n/2} \sigma_1(n, 1; j_1) \leq \|r\| \tilde{\rho}_n,$$

and

$$\sum_{|j_1 - n| > n/2} \frac{r(m + j_1)}{|n - j_1|} \sigma_1(n, 1; j_1) \leq \sum_{|j_1 - n| > n/2} \frac{r(m + j_1)}{|n - j_1|} \cdot \|r\| \leq \frac{2\|r\|}{\sqrt{n}} \cdot \|r\|.$$

Thus, in view of (3.6), we have

$$(3.9) \quad \sigma_1(n, 2; m) \leq 2\|r\| \cdot \tilde{\rho}_n.$$

On the other hand, for every $s \in \mathbb{N}$, we have

$$\sigma_1(n, s+2; m) = \sum_{j_1, \dots, j_s \neq n} \frac{r(m + j_1)}{|n - j_1|} \cdots \frac{r(j_{s-1} + j_s)}{|n - j_s|} \sum_{j_{s+1}, j_{s+2} \neq n} \frac{r(j_s + j_{s+1})}{|n - j_{s+1}|} \frac{r(j_{s+1} + j_{s+2})}{|n - j_{s+2}|}$$

$$= \sigma_1(n, s; m) \cdot \sup_{j_s} \sigma_1(n, 2; j_s).$$

Thus, by (3.9),

$$(3.10) \quad \sigma_1(n, s+2; m) \leq \sigma_1(n, s; m) \cdot 2\|r\|\tilde{\rho}_n.$$

Now it is easy to see, by induction in p , that (3.7), (3.9) and (3.10) imply (3.8). \square

Lemma 7. *Let $r = (r(k)) \in \ell^2(2\mathbb{Z})$ be the sequence defined by (2.12), and let*

$$(3.11) \quad \sigma_2(n, s; m) = \sum_{j_1, \dots, j_s \neq n} r(m+j_1) \frac{r(j_1+j_2)}{|n+j_2|} \dots \frac{r(j_{s-2}+j_{s-1})}{|n+j_{s-1}|} \frac{r(j_{s-1}+j_s)}{|n^2-j_s^2|}, \quad n \in \mathbb{N}, s \geq 2.$$

where $m, j_1, \dots, j_s \in n + 2\mathbb{Z}$. Then we have

$$(3.12) \quad \sigma_2(n, 2; m) \leq \|r\|^2 \cdot \frac{2 \log 6n}{n}$$

and

$$(3.13) \quad \sigma_2(n, s; m) \leq \|r\|^2 \cdot \frac{2 \log 6n}{n} \cdot \sup_k \sigma_1(n, s-2; k), \quad s \geq 3.$$

Proof. We have

$$(3.14) \quad \sigma_2(n, 2, m) = \sum_{j_2 \neq \pm n} \frac{1}{|n^2 - j_2^2|} \sum_{j_1 \neq \pm n} r(m+j_1) r(j_1+j_2).$$

By the Cauchy inequality, the sum over $j_1 \neq \pm n$ does not exceed $\|r\|^2$. Let us notice that

$$(3.15) \quad \sum_{j \neq \pm n} \frac{1}{n^2 - j^2} = \frac{2}{n} \sum_1^{2n} \frac{1}{k} - \frac{1}{2n^2} < \frac{2 \log 6n}{n}.$$

Therefore, (3.14) and (3.15) imply (3.12).

If $s \geq 3$ then the sum $\sigma_2(n, s; m)$ can be written in the form

$$\sigma_2(n, s; m) = \sum_{j_s \neq \pm n} \frac{1}{|n^2 - j_s^2|} \sum_{j_2, \dots, j_{s-1} \neq \pm n} \frac{r(j_2+j_3)}{|n+j_2|} \dots \frac{r(j_{s-1}+j_s)}{|n+j_{s-1}|} \sum_{j_1 \neq \pm n} r(m+j_1) r(j_1+j_2).$$

Changing the sign of all indices, one can easily see that the middle sum (over j_2, \dots, j_{s-1}) equals $\sigma_1(n; s-2, j_s)$. Thus, we have

$$\sigma_2(n, s; m) \leq \sum_{j_s \neq \pm n} \frac{1}{|n^2 - j_s^2|} \sigma_1(n; s-2, j_s) \cdot \sup_{j_2} \sum_{j_1 \neq \pm n} r(m+j_1) r(j_1+j_2).$$

By the Cauchy inequality, the sum over $j_1 \neq \pm n$ does not exceed $\|r\|^2$.

Therefore, by (3.15), we get (3.13). \square

Proof of Lemma 4. We set

$$(3.16) \quad \varepsilon_n = M \cdot \left[\left(\frac{2 \log 6n}{n} \right)^{1/4} + (\tilde{\rho}_n)^{1/2} \right],$$

where $M = 4(1 + \|r\|)$ is chosen so that for large enough n

$$(3.17) \quad \sup_m \sigma_1(n, 2p, m) \leq (\varepsilon_n/2)^{2p}, \quad \sup_m \sigma_1(n, 2p+1, m) \leq \|r\|(\varepsilon_n/2)^{2p}.$$

Then, for large enough n , we have

$$(3.18) \quad \sup_m \sigma_2(n, s, m) \leq \frac{1}{M}(\varepsilon_n/2)^{s+1}.$$

Indeed, by the choice of M , we have

$$(3.19) \quad \|r\|^2 \cdot \frac{2 \log 6n}{n} \leq \|r\|^2(\varepsilon_n/M)^4 \leq \frac{1}{M^2}(\varepsilon_n/2)^4.$$

Since $\varepsilon_n \rightarrow 0$, there is n_0 such that $\varepsilon_n < 1$ for $n \geq n_0$. Therefore, if $n \geq n_0$, then (3.12) and (3.19) yields (3.18) for $s = 2$. If $s = 2p$ with $p > 1$, then (3.13), (3.19) and (3.17) imply, for $n \geq n_0$,

$$\sup_m \sigma_2(n, 2p, m) \leq \frac{1}{M^2}(\varepsilon_n/2)^4 \cdot (\varepsilon_n/2)^{2p-2} \leq \frac{1}{M}(\varepsilon_n/2)^{2p+1}.$$

In an analogous way, for $n \geq n_0$, we get

$$\sup_m \sigma_2(n, 2p+1, m) \leq \frac{1}{M^2}(\varepsilon_n/2)^4 \cdot \|r\|(\varepsilon_n/2)^{2(p-1)} \leq \frac{1}{M}(\varepsilon_n/2)^{2p+2},$$

which completes the proof of (3.18).

Next we estimate $\sigma(n, s)$ by induction in s . By (3.7), we have for $n \geq n_0$,

$$(3.20) \quad \sigma(n, 1) = \sum_{j_1 \neq \pm n} \frac{r(n-j_1)}{|n-j_1|} = \sigma_1(n, 1; n) \leq \tilde{\rho}_n \leq (\varepsilon_n/2)^2 \leq \varepsilon_n.$$

For $s = 2$ we get, in view of (3.17) and (3.17):

$$\begin{aligned} (3.21) \quad \sigma(n, 2) &= \sum_{j_1, j_2 \neq \pm n} \left(\frac{1}{|n-j_1|} + \frac{1}{|n+j_2|} \right) \frac{1}{|n-j_2|} r(n+j_1) r(j_1+j_2) \\ &\leq \sum_{j_1, j_2 \neq \pm n} \frac{1}{|n-j_1|} \cdot \frac{1}{|n-j_2|} r(n+j_1) r(j_1+j_2) + \sum_{j_1, j_2 \neq \pm n} \frac{1}{|n+j_2|} \cdot \frac{1}{|n-j_2|} r(n+j_1) r(j_1+j_2) \\ &= \sigma_1(n, 2, n) + \sigma_2(n, 2, n) \leq (\varepsilon_n/2)^2 + (\varepsilon_n/2)^2 \leq (\varepsilon_n)^2. \end{aligned}$$

Next we estimate $\sigma(n, s)$, $s \geq 2$, Recall that $\sigma(n, s)$ is the sum of terms of the form

$$\Pi(j_1, \dots, j_s) r(n + j_1) r(j_1 + j_2) \cdots r(j_{s-1} + j_s),$$

where

(3.22)

$$\Pi(j_1, \dots, j_s) = \left(\frac{1}{|n - j_1|} + \frac{1}{|n + j_2|} \right) \cdots \left(\frac{1}{|n - j_{s-1}|} + \frac{1}{|n + j_s|} \right) \frac{1}{|n - j_s|}.$$

By opening the parentheses we get

(3.23)

$$\Pi(j_1, \dots, j_s) = \sum_{\delta_1, \dots, \delta_{s-1} = \pm 1} \left(\prod_{\nu=1}^{s-1} \frac{1}{|n + \delta_\nu j_\nu + \tilde{\delta}_\nu|} \right) \frac{1}{|n - j_s|}, \quad \tilde{\delta}_\nu = \frac{1 + \delta_\nu}{2}.$$

Therefore,

(3.24)

$$\sigma(n, s) = \sum_{\delta_1, \dots, \delta_{s-1} = \pm 1} \tilde{\sigma}(\delta_1, \dots, \delta_{s-1}),$$

where

(3.25)

$$\tilde{\sigma}(\delta_1, \dots, \delta_{s-1}) = \sum_{j_1, \dots, j_s \neq \pm n} \left(\prod_{\nu=1}^{s-1} \frac{1}{|n + \delta_\nu j_\nu + \tilde{\delta}_\nu|} \right) \frac{1}{|n - j_s|} r(n + j_1) r(j_1 + j_2) \cdots r(j_{s-1} + j_s).$$

In view of (2.49), (3.1) and (3.24), Lemma 4 will be proved if we show that

(3.26)

$$\tilde{\sigma}(\delta_1, \dots, \delta_{s-1}) \leq (\varepsilon_n/2)^s, \quad s \geq 2.$$

We prove (3.26) by induction in s .

If $s = 2$ then

$$\tilde{\sigma}(-1) = \sigma_1(n, 2, n) \leq (\varepsilon_n/2)^2,$$

and

$$\tilde{\sigma}(+1) = \sigma_2(n, 2, n) \leq (\varepsilon_n/2)^2.$$

If $s = 3$ then there are four cases:

$$\tilde{\sigma}(-1, -1) = \sigma_1(n, 3, n) \leq (\varepsilon_n/2)^3; \quad \tilde{\sigma}(+1, +1) = \sigma_2(n, 3, n) \leq (\varepsilon_n/2)^3;$$

$$\begin{aligned} \tilde{\sigma}(-1, +1) &= \sum_{j_1 \neq \pm n} \frac{r(n + j_1)}{|n - j_1|} \sum_{j_2, j_3 \neq \pm n} \frac{r(j_1 + j_2)}{|n + j_3|} \frac{r(j_2 + j_3)}{|n - j_3|} \\ &= \sum_{j_1 \neq \pm n} \frac{r(n + j_1)}{|n - j_1|} \sigma_2(n, 2, j_1) \\ &\leq \sigma_1(n, 1, n) \cdot \sup_m \sigma_2(n, 2, m) \leq \|r\| \frac{1}{K} (\varepsilon_n/2)^3 \leq (\varepsilon_n/2)^3; \end{aligned}$$

$$\begin{aligned}
\tilde{\sigma}(+1, -1) &= \sum_{j_1, j_2 \neq \pm n} \frac{r(n+j_1)r(j_1+j_2)}{|n^2-j_2^2|} \sum_{j_3 \neq \pm n} \frac{r(j_2+j_3)}{|n-j_3|} \\
&\leq \sigma_2(n, 2, n) \cdot \sup_m \sigma_1(n, 1, m) \leq \frac{1}{K}(\varepsilon_n/2)^3 \|r\| \leq (\varepsilon_n/2)^3.
\end{aligned}$$

Next we prove that if (3.26) hold for some s , then it holds for $s+2$. Indeed, let us consider the following cases:

(i) $\delta_s = \delta_{s+1} = -1$; then we have

$$\begin{aligned}
\tilde{\sigma}(\delta_1, \dots, \delta_{s-1}, -1, -1) &= \sum_{j_1, \dots, j_s \neq \pm n} \left(\prod_{\nu=1}^{s-1} \frac{1}{|n + \delta_\nu j_\nu + \tilde{\delta}_\nu|} \right) \frac{1}{|n - j_s|} \\
&\times r(n+j_1)r(j_1+j_2) \cdots r(j_{s-1}+j_s) \sum_{j_{s+1}, j_{s+2} \neq \pm n} \frac{r(j_s+j_{s+1})}{|n-j_{s+1}|} \frac{r(j_{s+1}+j_{s+2})}{|n-j_{s+2}|} \\
&= \sum_{j_1, \dots, j_s \neq \pm n} \left(\prod_{\nu=1}^{s-1} \frac{1}{|n + \delta_\nu j_\nu + \tilde{\delta}_\nu|} \right) \frac{1}{|n - j_s|} r(n+j_1) \cdots r(j_{s-1}+j_s) \sigma_1(n, 2, j_s) \\
&\leq \tilde{\sigma}(\delta_1, \dots, \delta_{s-1}) \cdot \sup_m \sigma_1(n, 2, m) \leq (\varepsilon_n/2)^s \cdot (\varepsilon_n/2)^2 = (\varepsilon_n/2)^{s+2}.
\end{aligned}$$

(ii) $\delta_s = -1, \delta_{s+1} = +1$; then we have

$$\begin{aligned}
\tilde{\sigma}(\delta_1, \dots, \delta_{s-1}, -1, +1) &= \sum_{j_1, \dots, j_s \neq \pm n} \left(\prod_{\nu=1}^{s-1} \frac{1}{|n + \delta_\nu j_\nu + \tilde{\delta}_\nu|} \right) \frac{1}{|n - j_s|} \\
&\times r(n+j_1)r(j_1+j_2) \cdots r(j_{s-1}+j_s) \sum_{j_{s+1}, j_{s+2} \neq \pm n} \frac{r(j_s+j_{s+1})r(j_{s+1}+j_{s+2})}{|n^2-j_{s+2}^2|} \\
&\leq \tilde{\sigma}(\delta_1, \dots, \delta_{s-1}) \cdot \sup_m \sigma_2(n, 2, m) \leq (\varepsilon_n/2)^s \cdot (\varepsilon_n/2)^2 = (\varepsilon_n/2)^{s+2}.
\end{aligned}$$

(iii) $\delta_s = \delta_{s+1} = +1$; then, if $\delta_1 = \cdots = \delta_{s-1} = +1$, we have

$$\tilde{\sigma}(\delta_1, \dots, \delta_{s+1}) = \sigma_2(n, s+2, n) \leq (\varepsilon_n/2)^{s+2}.$$

Otherwise, let $\mu < s$ be the largest index such that $\delta_\mu = -1$. Then we have

$$\begin{aligned}
\tilde{\sigma}(\delta_1, \dots, \delta_{s-1}, +1, +1) &= \sum_{j_1, \dots, j_\mu \neq \pm n} \left(\prod_{\nu=1}^{\mu-1} \frac{1}{|n + \delta_\nu j_\nu + \tilde{\delta}_\nu|} \right) \frac{1}{|n - j_\mu|} \\
&\times r(n+j_1)r(j_1+j_2) \cdots r(j_{\mu-1}+j_\mu) \sigma_2(n, s+2-\mu, j_\mu) \\
&\leq \tilde{\sigma}(\delta_1, \dots, \delta_{\mu-1}) \cdot \sup_m \sigma_2(n, s+2-\mu, j_\mu) \leq (\varepsilon_n/2)^\mu \cdot (\varepsilon_n/2)^{s+2-\mu} = (\varepsilon_n/2)^{s+2}.
\end{aligned}$$

(iv) $\delta_s = +1, \delta_{s+1} = -1$; then, if $\delta_1 = \cdots = \delta_{s-1} = +1$, we have

$$\tilde{\sigma}(\delta_1, \dots, \delta_{s+1}, -1) = \tilde{\sigma}(+1, \dots, +1, -1, -1) =$$

$$\begin{aligned}
&= \sum_{j_1, \dots, j_{s+1} \neq \pm n} \left(\prod_{\nu=1}^s \frac{1}{|n + j_{\nu+1}|} \right) \frac{1}{|n - j_{s+1}|} r(n + j_1) \cdots r(j_s + j_{s+1}) \sigma_1(n, 1, j_{s+1}) \\
&\leq \sigma_2(n, s + 1, n) \cdot \sup_m \sigma_1(n, 1, m) \leq \frac{1}{K} (\varepsilon_n/2)^{s+2} \cdot \|r\| \leq (\varepsilon_n/2)^{s+2}.
\end{aligned}$$

Otherwise, let $\mu < s$ be the largest index such that $\delta_\mu = -1$, $1 \leq \mu < n$. Then we have

$$\begin{aligned}
&\tilde{\sigma}(\delta_1, \dots, \delta_{s-1}, +1, -1) = \sum_{j_1, \dots, j_\mu \neq \pm n} \left(\prod_{\nu=1}^{\mu-1} \frac{1}{|n + \delta_\nu j_\nu + \delta_\nu|} \right) \frac{1}{|n - j_\mu|} \\
&\times \sum_{j_{\mu+1}, \dots, j_{s+1} \neq \pm n} \frac{r(j_\mu + j_{\mu+1})}{|n + j_{\mu+2}|} \cdots \frac{r(j_{s-1} + j_s)}{|n + j_{s+1}|} \frac{r(j_s + j_{s+1})}{|n - j_{s+1}|} \sum_{j_{s+2} \neq \pm n} \frac{r(j_{s+1} + j_{s+2})}{|n - j_{s+2}|} \\
&\leq \tilde{\sigma}(\delta_1, \dots, \delta_{\mu-1}) \cdot \sup_m \sigma_2(n, s + 1 - \mu, m) \cdot \sup_k \sigma_1(n, 1, k) \\
&\leq (\varepsilon_n/2)^\mu \cdot \frac{1}{K} (\varepsilon_n/2)^{s+2-\mu} \|r\| \leq (\varepsilon_n/2)^{s+2}.
\end{aligned}$$

Hence (3.26) holds for $s \geq 2$.

Now (3.1), (3.24) and (3.26) imply (2.49), which completes the proof of Lemma 4. \square

Now we are ready to accomplish the proof of Theorem 2.

4. PROOF OF THE MAIN THEOREM

We need – because we want to use (2.8) – to give estimates of $A(n, s)$ from (2.9), or (2.20). By (2.25) and (2.26), we reduce such estimates to analysis of quantities $A_j(n, s)$, $j = 1, \dots, 7$.

With $\rho_n \in (2.34)$ and $\varepsilon_n \in (3.16)$, we set

$$(4.1) \quad \kappa_n = \max\{\rho_n, \varepsilon_n\}.$$

Then, by Lemma 4 (and Corollary 5), i.e., by the inequality (2.50), we have (in view of (2.36), (2.39), (2.42) and (2.44)–(2.47)) the following estimates for A_j :

$$\begin{aligned}
A_1 &\leq 4\kappa_n^{s+1}, \quad A_j \leq 2^{s+1} \cdot 2\kappa_n^{s+1}, \quad j = 2, 4; \\
A_j &\leq 2^{s+1} \sum_{\nu=1}^s (2\kappa_n^\nu \cdot \kappa_n^{s-\nu+1}) = s2^{s+2} \kappa_n^{s+1}, \quad j = 3, 5; \\
A_6 &\leq 2^{s+1} \sum_{\nu=1}^s (\kappa_n^\nu \cdot \kappa_n^{s-\nu+1}) = s2^{s+1} \kappa_n^{s+1}; \\
A_7 &\leq 2^s \sum_{1 \leq \nu + \mu \leq s} (4\kappa_n^\nu \cdot \kappa_n^{\mu-\nu} \cdot \kappa_n^{s-\mu+1}) = s(s-1)2^{s+1} \kappa_n^{s+1}.
\end{aligned}$$

In view of (2.16), (3.16) and (2.26), these inequalities imply

$$A(n, s) \leq (2 + s)^2 (2\kappa_n)^{s+1}.$$

Therefore, the right-hand side of (2.8) does not exceed

$$\sum_{s=0}^{\infty} A(n, s) \leq (4\kappa_n) \sum_{s=0}^{\infty} (s+1)(s+2)(2\kappa_n)^s = \frac{8\kappa_n}{(1-2\kappa_n)^3}.$$

Therefore, if $\kappa_n < 1/4$ (which holds for $n \geq N^*$ with a proper choice of N^*), then $\sum_{s=0}^{\infty} A(n, s) \leq 64\kappa_n$. Thus, by (2.8) and the notations (2.3),

$$(4.2) \quad \|P_n - P_n^0\|_{L^1 \rightarrow L^\infty} \leq \sum_{k,m} |B_{km}(n)| \leq 64\kappa_n, \quad n \geq N^*,$$

where $\kappa_n \in (4.1)$.

This completes the proof of Theorem 2. Of course, Proposition 1 follows because $\|T\|_{L^2 \rightarrow L^2} \leq \|T\|_{L^1 \rightarrow L^\infty}$ for any well defined operator T . \square

5. MISCELLANEOUS

1. Theorem 2 (or Proposition 1) is an essential step in the proof of our general statement (see an announcement in [6], Thm. 9, or [7], Thm. 23), about the relationship between the rate of decay of spectral gap sequences (and deviations) and the smoothness of the potentials v under the *a priori* assumption that v is a singular potential, i.e., that $v \in H_{Per}^{-1}$. To use the information about the deviations $\delta_n = |\mu_n - \frac{1}{2}(\lambda_n^+ + \lambda_n^-)|$, this is done in the framework of the scheme suggested by the authors in [2]. The concluding steps will be presented in an upcoming paper, the third after [6] and the present one. However, Theorem 2 is important outside this context as well. We will mention now the most obvious corollaries.

2. The following theorem holds.

Theorem 8. *In the above notations, the L^p -norms, $1 \leq p \leq \infty$, on Riesz subspaces $E^N = \text{Ran } S_N$, and $E_n = \text{Ran } P_n$, $n \geq N$, are uniformly equivalent; more precisely,*

$$(5.1) \quad \|f\|_1 \leq \|f\|_\infty \leq C(N)\|f\|_1, \quad \forall f \in E^N,$$

and

$$(5.2) \quad \|f\|_\infty \leq 3\|f\|_1, \quad \forall f \in E_n, \quad n \geq N^*(v),$$

where

$$(5.3) \quad C(N) \leq 50N \ln N.$$

Proof. By (2.2), if N is large enough,

$$(5.4) \quad \|P_n - P_n^0\|_{L^1 \rightarrow L^\infty} \leq \frac{1}{2}, \quad n \geq N.$$

If we are more careful when using (4.1), (4.2), (2.34) and (3.16), we may claim (5.4) for N such that

$$(5.5) \quad 2^9(1 + \|r\|) \left(\mathcal{E}_{\sqrt{N}}(r) + \frac{2}{N^{1/4}} (\|r\|^{1/2} + (\ln 6N)^{1/4}) \right) \leq \frac{1}{2}.$$

If $f \in E_n$, $n \geq N$, we have

$$(5.6) \quad f = P_n f = (P_n - P_n^0)f + P_n^0 f,$$

where, for $bc = Per^\pm$

$$(5.7) \quad P_n^0 f = f_n e^{inx} + f_{-n} e^{-inx}, \quad f_k = \frac{1}{\pi} \int_0^\pi f(x) e^{-ikx} dx,$$

and, for $bc = Dir$,

$$(5.8) \quad P_n^0 f = 2g_n \sin nx, \quad g_n = \frac{1}{\pi} \int_0^\pi f(x) \sin nx dx.$$

In either case $\|P_n f\|_\infty \leq 2\|f\|_1$, and therefore, if $\|f\|_1 \leq 1$ we have

$$(5.9) \quad \|f\|_\infty \leq \|(P_n - P_n^0)f\|_\infty + \|P_n^0 f\|_\infty \leq 1/2 + 2 \leq 3.$$

Remind that a projection

$$(5.10) \quad S_N = \frac{1}{2\pi i} \int_{\partial R_N} (z - L_{bc})^{-1} dz,$$

where, as in (5.40), [7],

$$(5.11) \quad R_N = \{z \in \mathbb{C} : -N < \operatorname{Re} z < N^2 + N, |\operatorname{Im} z| < N,$$

is finite-dimensional (see [7], (5.54), (5.56), (5.57) for $\dim S_N$). Now we follow the inequalities proven in [7] to explain (5.1) and (5.3). Lemma 20, inequality (5.41) in [7], states that

$$(5.12) \quad \sup\{\|K_\lambda V K_\lambda\|_{HS} : \lambda \notin R_N, \operatorname{Re} \lambda \leq N^2 - N\} \leq C \left(\frac{(\log N)^{1/2}}{N^{1/4}} \|q\| + \mathcal{E}_{4\sqrt{N}}(q) \right).$$

But by (5.10)

$$(5.13) \quad S_N - S_N^0 = \frac{1}{2\pi i} \int_\Gamma K_\lambda \sum_{m=1}^\infty (K_\lambda V K_\lambda)^m K_\lambda d\lambda,$$

where we can choose Γ to be the boundary $\partial\Pi$ of the rectangle

$$(5.14) \quad \Pi(H) = \{z \in \mathbb{C} : -H \leq \operatorname{Re} z \leq N^2 + N, |\operatorname{Im} z| \leq H\}, \quad H \geq N.$$

Then by (5.12) and (5.13) the norm of the sum in the integrand can be estimated by

$$(5.15) \quad \left\| \sum_1^\infty \right\|_{2 \rightarrow 2} \leq \sum_1^\infty \|K_\lambda V K_\lambda\|_{HS}^m \leq 1, \quad \forall \lambda \in \partial\Pi(H)$$

if (compare with (5.5)) $N \geq N^*(q)$ and $N^* = N^*(q)$ is chosen to guarantee that

$$(5.16) \quad \text{“the right side in (5.12)”} \leq 1/2 \text{ for } N \geq N^*.$$

The additional factor K_λ is a multiplier operator defined by the sequence $\tilde{K} = \{1/\sqrt{\lambda - k^2}\}$, so its norms $\|K_\lambda : L^1 \rightarrow L^2\|$ and $\|K_\lambda : L^1 \rightarrow L^2\|$ are estimated by $2\tilde{\kappa}$, where

$$(5.17) \quad \tilde{\kappa} = \|\tilde{K}_\lambda : \ell^\infty \rightarrow \ell^2\| = \|\tilde{K}_\lambda : \ell^2 \rightarrow \ell^1\| = \sum_k \frac{1}{|\lambda - k^2|}.$$

Therefore, by (5.15) and (5.17),

$$(5.18) \quad \alpha(\lambda) := \|K_\lambda \left(\sum_1^\infty \cdots \right) K_\lambda : L^1 \rightarrow L^\infty\| \leq \sum_k \frac{4}{|\lambda - k^2|}.$$

By Lemma 18(a) in [7] (or, Lemma 79(a) in [5])

$$(5.19) \quad \sum_k \frac{1}{|n^2 - k^2| + b} \leq C_1 \frac{\log b}{\sqrt{b}} \quad \text{if } n \in \mathbb{N}, b \geq 2.$$

(In what follows C_j , $j = 1, 2, \dots$ are absolute constants; $C_1 \leq 12$.) These inequalities are used to estimate the norm $\alpha(\lambda)$ on the boundary $\partial\Pi(H) = \cup I_k(H)$, $k = 1, 2, 3, 4$, where

$$\begin{aligned} I_1(H) &= \{z : \operatorname{Re} z = -H, |\operatorname{Im} z| \leq H\} \\ I_2(H) &= \{z : -H \leq \operatorname{Re} z \leq N^2 + N, \operatorname{Im} z = H\} \\ I_3(H) &= \{z : \operatorname{Re} z = N^2 + N, |\operatorname{Im} z| \leq H\} \\ I_4(H) &= \{z : -H \leq \operatorname{Re} z \leq N^2 + N, \operatorname{Im} z = -H\} \end{aligned}$$

Then we get

$$\begin{aligned} \int_{I_1} \alpha(\lambda) |d\lambda| &\leq C_2 \frac{\log H}{\sqrt{H}} \cdot H, \\ \int_{I_k} \alpha(\lambda) |d\lambda| &\leq C_3 \frac{\log H}{\sqrt{H}} \cdot N^2, \quad k = 2, 4. \\ \int_{I_3} \alpha(\lambda) |d\lambda| &\leq C_4 \int_0^H \frac{\log(N + y)}{\sqrt{N + y}} dy \leq C_5 \sqrt{H} \log H. \end{aligned}$$

If we put $H = N^2$ and sum up these inequalities we get by (5.13)

$$(5.20) \quad \|S_N - S_N^0\|_{L^1 \rightarrow L^\infty} \leq C_6 N \log N,$$

where C_6 is an absolute constant ≤ 600 .

Now, as in (5.6) and (5.7), let us notice that for $g \in E^N$

$$(5.21) \quad g = S_N g = (S_N - S_N^0)g + S_N^0 g,$$

where

$$(5.22) \quad S_N^0 g = \sum_{|k| \leq N} g_k e^{ikx}, \quad k \text{ even for } bc = Per^+, \text{ odd for } bc = Per^-,$$

and

$$(5.23) \quad S_N^0 g = 2 \sum_{|k| \leq N} \tilde{g}_k \sin kx, \quad bc = Dir,$$

where

$$(5.24) \quad g_k = \frac{1}{\pi} \int_0^\pi g(x) e^{ikx} dx, \quad \tilde{g}_k = \frac{1}{\pi} \int_0^\pi g(x) \sin kx dx.$$

In either case

$$(5.25) \quad \|S_N^0 g\|_\infty \leq 2N \|g\|_1.$$

Therefore, by (5.20) and (5.25), if $\|f\|_1 \leq 1$ we have

$$(5.26) \quad \|f\|_\infty \leq C_6 N \log N + 2N \leq C_7 N \log N, \quad N \geq N^* \in (5.16).$$

Let us fix $N_0 \geq N^*, N_*$, where N_* is determined by (5.5), i.e., (5.5) holds if $N \geq N_*$. Then, by (5.26),

$$(5.27) \quad \|S_{N_0}\|_{L^1 \rightarrow L^\infty} \leq C N_0 \log N_0,$$

and for $N > N_0$ we may improve the estimate in (5.26). Indeed,

$$S_N = (S_N - S_{N_0}) + S_{N_0} = S_{N_0} + \sum_{N_0+1}^N P_k$$

and, by (5.4) and (5.7), $\|P_k\|_{L^1 \rightarrow L^\infty} \leq 3$. Therefore, by (5.27),

$$\|S_N\|_{L^1 \rightarrow L^\infty} \leq C N_0 \log N_0 + (N - N_0) \leq 3N + C N_0 \log N_0.$$

□

3. Of course, any estimates of the kind

$$(5.28) \quad \|S_N - S_{N_0}\|_{L^1 \rightarrow L^\infty} \leq C(N)$$

with $C_N \rightarrow \infty$ as $N \rightarrow \infty$ are weaker than the claim

$$(5.29) \quad \omega_N = \|S_N - S_{N_0}\|_{L^1 \rightarrow L^\infty} \rightarrow 0$$

or even that ω_N is a bounded sequence. For real-valued potentials $v \in H^{-1}$ and $bc = Dir$, (5.29) would follow from Theorem 1 in [12] if its proof given in [12] were valid. For complex-valued potentials $v \in H^{-1}$, when the system of eigenfunctions is not necessarily orthogonal the statement of Theorem 1

in [12] is false. Maybe it could be corrected if the "Fourier coefficients" are chosen as

$$c_k(f) = \langle f, w_k \rangle$$

where the system $\{w_k\}$ is bi-orthonormal with respect to $\{u_k\}$, i.e.,

$$\langle u_j, w_k \rangle = \delta_{jk}$$

(not the way as it is done in [12]). But more serious oversight, not just a technical misstep, seems to be a crucial reference to [13], without specifying lines or statements in [13], to claim something that cannot be found there. Namely, the author of [12] alleges that in [13] the following statement is proven. *Let $\{y_k(x)\}$ be a normalized system of eigenfunctions of the operator*

$$L = -d^2/dx^2 + v, \quad v \in H^{-1}([0, \pi]),$$

considered with Dirichlet boundary conditions. Then

$$(5.30) \quad y_k(x) = \sqrt{2} \sin kx + \psi_k(x),$$

where

$$(5.31) \quad \sup_{[0, \pi]} |\psi_k(x)| \in \ell^2.$$

(Two more sup-sequences coming from derivatives y'_k are claimed to be in ℓ^2 as well.)

However, what one could find in [13], Theorem 2.7 and Theorem 3.13(iv),(v), is the claim

$$(5.32) \quad \sup_{[0, \pi]} \sum_k |\psi_k(x)|^2 < \infty.$$

Of course, (5.31) implies (5.32) but if $\{\psi_k\}$ is a sequence of L^∞ -functions then (5.32) does not imply (5.31).

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